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KREIN'S STRINGS WHOSE SPECTRAL FUNCTIONS ARE OF POLYNOMIAL GROWTH

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Abstract. In the case of Krein's strings with spectral functions of polynomial growth a necessary and sufficient condition for the Krein's correspondence to be continuous is given.

1. Introduction

Let $\mathcal{M}$ be the totality of non-decreasing, right continuous functions on $[0,\infty)$ satisfying

$$m(0-)=0,\quad m(x)\leq\infty,$$

and set

$$\begin{cases} l=\inf\left\{x\geq 0;\ m(x)=\infty\right\}, \\ a=\inf\left\{x\geq 0;\ m(x)>0\right\}. \end{cases}$$

For $m\in\mathcal{M}$ denote $\varphi_\lambda(x),\psi_\lambda(x)$ the solutions to

$$\begin{cases} \varphi_\lambda(x)=1-\lambda\displaystyle\int_0^x(x-y)\,\varphi_\lambda(y)\,dm(y), \\ \psi_\lambda(x)=x-\lambda\displaystyle\int_0^x(x-y)\,\psi_\lambda(y)\,dm(y), \end{cases}$$

and define

$$h(\lambda)=\lim_{x\to l}\frac{\psi_\lambda(x)}{\varphi_\lambda(x)}=\int_0^l\varphi_\lambda(x)^{-2}\,dx.$$

Then it is known that there exists a unique measure σ on $[0,\infty)$ satisfying

$$h(\lambda)=a+\int_0^\infty\frac{1}{\xi-\lambda}d\sigma(\xi),$$

and conversely, h determines m uniquely. Conventionally it is understood that for $m\in\mathcal{M}$ taking ∞ identically on $[0,\infty)$ the h vanishing identically corresponds, and for $m\in\mathcal{M}$ vanishing identically on $[0,\infty)$ the h taking identically ∞ corresponds. This is the theorem obtained by Krein[8] and m is called Krein's (regular) string. Later Kasahara[1] established the continuity for the correspondence and applied it to show limit theorems for 1D diffusion processes with m their speed measures. Recently Kotani[7] extended Kasahara's result to a certain kind of singular strings m, namely to m which is a non-decreasing and right continuous function on $(-\infty,\infty)$ satisfying

$$m(-\infty)=0,\quad m(x)\leq\infty,$$

and

$$\int_{-\infty}^a x^2dm(x)<\infty \tag{1.1}$$

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for some a. When the condition (1.1) is satisfied, the boundary $-\infty$ is called as the limit circle type for the associated generalized second order differential operator $d^2/dmdx$. In this case he introduced a new h by

$$h(\lambda) = \lim_{x\to-\infty}\left(x + \varphi_\lambda(x)\int_x^l \frac{dy}{\varphi_\lambda(y)^2}\right) = a + \int_0^\infty \left(\frac{1}{\xi-\lambda} - \frac{\xi}{\xi^2+1}\right) d\sigma(\xi),$$

which satisfies

$$h'(\lambda) = \int_{-\infty}^l \frac{\partial}{\partial\lambda}\varphi_\lambda(x)^{-2}\,dx,$$

and proved the continuity of the correspondence between m and h. Probabilistic applications of this result were given by Kasahara-Watanabe[2, 3] and it was interpreted from the point of view of the excursion theory by Yano[9]. In this article we consider m satisfying a milder condition than (1.1), namely

$$\int_{-\infty}^a |x|\,dm(x) < \infty,$$

and obtain the continuity result under additional conditions on m, which allows any power growth of the spectral measures at ∞.

2. Preliminaries

Let $m(x)$ be a non-decreasing and right continuous function on $(-\infty,\infty)$ satisfying

$$m(-\infty) = 0, \qquad m(\infty) \le \infty.$$

Set

$$l = \sup\{x > -\infty,\ m(x) < +\infty\}, \quad l_+ = \sup \mathrm{supp} dm, \quad l_- = \inf \mathrm{supp} dm.$$

Note $m(l) = \infty$ if $l < \infty$. Assume

$$\int_{-\infty}^a |x|\,dm(x) < \infty \tag{2.1}$$

with some $a \in (l_-, l_+)$. Let $\mathcal{E}$ be the totality of non-decreasing functions m satisfying (2.1). We exclude m vanishing identically on $(-\infty,\infty)$ from $\mathcal{E}$. One can regard dm as a distribution of weight and in this case m works as a string. On the other hand, one can associate a generalized diffusion process with generator L

$$L = \frac{d}{dm}\frac{d}{dx}$$

if we impose a suitable boundary condition if necessary. The condition (2.1) is called as entrance condition in 1D diffusion theory developed by W.Feller, so we call m satisfying (2.1) a string of entrance type. For an entrance type m, it is easy to show that for $\lambda \in \mathbf{C}$ an integral equation

$$\varphi(x) = 1 - \lambda\int_{-\infty}^x (x-y)\varphi(y)dm(y)$$

has a unique solution, which is denoted by $\varphi_\lambda(x)$. Introduce a subspace

$$L_0^2(dm) = \left\{f \in L^2(dm);\ \mathrm{supp} f \subset (-\infty, l)\right\},$$

and for $f \in L_0^2(dm)$ define a generalized Fourier transform by

$$\widehat{f}(\lambda) = \int_{-\infty}^l f(x)\varphi_\lambda(x)dm(x).$$

Krein's spectral theory implies there exists a measure σ on $[0,\infty)$ satisfying

$$\int_{-\infty}^{l} |f(x)|^2\, dm(x) = \int_0^{\infty} \left|\widehat{f}(\xi)\right|^2 d\sigma(\xi) \quad \text{for any } f \in L_0^2(dm). \tag{2.2}$$

σ is called a spectral measure for the string m. The non-uniqueness of such σ occurs if and only if

$$l_+ + m\left(l_+\right) < \infty. \tag{2.3}$$

The number $l\,(\geq l_+)$ possesses its meaning only when (2.3) is satisfied, and in this case there exists a σ satisfying (2.2) with the boundary condition

$$f\left(l_+\right) + \left(l - l_+\right) f^+\left(l_+\right) = 0$$

at l_+. Here f^+ is the derivative from the right hand side. If $l = \infty$, this should be interpreted as

$$f^+\left(l_+\right) = 0.$$

At the left boundary l_- no boundary condition is necessary if $l_- = -\infty$, and if $l_- > -\infty$ we impose the reflective boundary condition, namely

$$f^-\left(l_-\right) = 0 \quad \text{the derivative from left.}$$

Generally, for a string m of entrance type it is known that for $\lambda < 0$ there exists uniquely f such that

$$\begin{cases} -Lf = \lambda f, \quad f > 0, \quad f^+ \leq 0, \quad f\left(l-\right) = 0 \\ f(x)\varphi_\lambda^+(x) - f^+(x)\varphi_\lambda(x) = 1 \end{cases}.$$

This unique f is denoted by f_λ and contains information of the boundary condition we are imposing on $-L$ at the right boundary l_+, and f_λ can be represented by φ_λ as

$$f_\lambda(x) = \varphi_\lambda(x) \int_x^l \frac{dy}{\varphi_\lambda(y)^2}. \tag{2.4}$$

The right side integral is always convergent for $\lambda < 0$, because if supp$dm \neq \phi$, then choosing $a \in$ suppdm, we see for $x > a$

$$\varphi_\lambda(x) \geq 1 - \lambda \int_{-\infty}^{x} (x-y)\, dm(y) \geq 1 - \lambda \int_{-\infty}^{a} (x-y)\, dm(y) \geq 1 - \lambda\,(x-a)\, m(a),$$

hence

$$\int_x^l \frac{dy}{\varphi_\lambda(y)^2} \leq \int_x^l \frac{dy}{\left(1 - \lambda\,(y-a)\, m(a)\right)^2} < \infty$$

for $x > a$. If supp$dm = \phi$, then $l < \infty$ and

$$m(x) = \begin{cases} 0 & \text{for} \quad x < l \\ \infty & \text{for} \quad x > l \end{cases}, \tag{2.5}$$

which implies

$$\varphi_\lambda(x) = \begin{cases} 1 & \text{for} \quad x < l \\ \infty & \text{for} \quad x > l \end{cases},$$

and

$$\int_x^l \frac{dy}{\varphi_\lambda(y)^2} = l - x < \infty.$$

Here note that we have excluded $m = 0$ identically on $(-\infty, \infty)$, hence $l < \infty$. If m is a non-decreasing function of (2.5) the spectral measure vanishes identically on $[0,\infty)$. If m is ∞ identically on $(-\infty,\infty)$, the spectral function σ is defined to be 0 identically on $[0,\infty)$. Conversely if a spectral measure vanishes identically on $[0,\infty)$, then the associated string m should be of (2.5). $\varphi_\lambda(x)$ is an entire function of minimal exponential type as a function of λ and the zeroes of $\varphi_\lambda(x)$ coincide

with the eigenvalues of $-L$ defined as a self-adjoint operator on $L^2(dm, (-\infty, x])$ with the Dirichlet boundary condition at x, which means that $\varphi_\lambda(x)$ has simple zeroes on $(0, \infty)$. The Green function g_λ for $-L$ on $L^2(dm)$ is given by

$$g_\lambda(x, y) = g_\lambda(y, x) = f_\lambda(y)\varphi_\lambda(x)$$

for $x \leq y$. The relationship between σ and g_λ is described by an identity

$$\int_{-\infty}^{l}\int_{-\infty}^{l} g_\lambda(x, y)f(x)\overline{f(y)}dm(x)dm(y) = \int_0^\infty \frac{\left|\widehat{f}(\xi)\right|^2}{\xi - \lambda}\sigma(d\xi)$$

for any $f \in L^2(dm)$, and

$$g_\lambda(x, y) = \int_0^\infty \frac{\varphi_\xi(x)\varphi_\xi(y)}{\xi - \lambda} d\sigma\left(\xi\right),$$

through which σ is determined uniquely from the string m. Distinct ms may give an equal σ, namely for $a \in \mathbf{R}$ a new string

$$m_a\left(x\right) = m\left(x + a\right)$$

defines the same σ, because

$$\varphi_\lambda^a(x) = \varphi_\lambda(x + a), \quad f_\lambda^a(x) = f_\lambda(x + a),$$

hence

$$g_\lambda^a(x, x) = \varphi_\lambda^a(x)f_\lambda^a(x) = g_\lambda(x + a, x + a) = \int_0^\infty \frac{\varphi_\xi(x + a)^2}{\xi - \lambda} d\sigma\left(\xi\right).$$

On the other hand

$$g_\lambda^a(x, x) = \int_0^\infty \frac{\varphi_\xi^a(x)\varphi_\xi^a(x)}{\xi - \lambda} d\sigma_a\left(\xi\right) = \int_0^\infty \frac{\varphi_\xi(x + a)^2}{\xi - \lambda} d\sigma_a\left(\xi\right),$$

hence an identity

$$\sigma_a\left(\xi\right) = \sigma\left(\xi\right)$$

should be held. Conversely we have

Theorem 1. (*Kotani*[5, 6]) *If two strings m_1 and m_2 of $\mathcal{E}$ have the same spectral measure σ, then $m_1(x + c) = m_2(x)$ for a $c \in \mathbf{R}$.*

If we hope to obtain the continuity of the correspondence between m and σ, we have to keep the non-uniqueness in mind. Namely, for m of $\mathcal{E}$ a sequence $\{m_n\}_{n \geq 1}$ of $\mathcal{E}$ defined by

$$m_n\left(x\right) = m\left(x - n\right)$$

converges to the trivial function 0 as $n \to \infty$. However the associated σs are independent of n. Therefore we shall give several alternative definitions of convergence by imposing certain extra conditions (related to tightness) in addition to pointwise convergence. Set

$$M(x) = \int_{-\infty}^{x} \left(x - y\right) dm(y) = \int_{-\infty}^{x} m(y)dy.$$

Then, the condition (2.1) is equivalent to

$$M(x) < \infty$$

for $x < l$. Using a convention

$$\left[-\infty, a\right) = \left(-\infty, a\right), \ \left(a, \infty\right] = \left(a, \infty\right) \text{ and so on,}$$

we can see M is a non-decreasing convex function on $(-\infty, \infty)$ satisfying

$$\begin{cases} M\left(x\right) = 0 \ \text{ on } \left(-\infty, l_-\right], \\ \text{continuous and strictly increasing on } \left[l_-, l\right), \\ M\left(x\right) = \infty \text{ on } \left(l, \infty\right). \end{cases}$$

For a fixed positive number c, we assume

$$0 \in (l_-, l] \text{ and } M(l) \geq c, \tag{2.6}$$

and normalize such an m by

$$M(0) = c. \tag{2.7}$$

Denote by $\mathcal{E}^{(c)}$ the set of all elements of $\mathcal{E}$ satisfying (2.6), (2.7) and set

$$\mathcal{E}_+ = \bigcup_{c>0} \mathcal{E}^{(c)}.$$

In this definition of $\mathcal{E}_+$ among functions satisfying (2.1) any function m defined by (2.5) for some $l \leq \infty$ is excluded from $\mathcal{E}_+$. Therefore, $\mathcal{E}\backslash\mathcal{E}_+$ consists of m satisfying (2.5) for some $l < \infty$. The uniqueness of the correspondence between m and σ holds under this normalization. Set

$$\mathcal{S} = \text{the set of all spectral measures for strings of } \mathcal{E}.$$

Any suitable characterization of $\mathcal{S}$ is not known yet, however, any measure on $[0, \infty)$ with polynomial growth at ∞ belongs to $\mathcal{S}$.

We prepare a basic estimate for φ_λ. φ_λ can be represented as

$$\varphi_\lambda(x) = \sum_{n=0}^{\infty} (-\lambda)^n \phi_n(x), \tag{2.8}$$

where $\{\phi_n\}_{n\geq 0}$ are

$$\phi_n(x) = \int_{-\infty}^{x} (x-y)\, \phi_{n-1}(y) dm(y), \qquad \phi_0(x) = 1.$$

Then, the convergence of the above series can be seen by

Lemma 1. *φ_λ is given by an absolute convergent series (2.8) and satisfies*

$$|\varphi_\lambda(x)| \leq \exp(|\lambda| M(x)).$$

Proof. First we show for any $k \geq 0$

$$\phi_k(x) \leq \frac{M(x)^k}{k!} \tag{2.9}$$

holds. Observe

$$\phi_1(x) = \int_{-\infty}^{x} (x-y)\, dm(y) = M(x).$$

Assuming (2.9) for some k, we have

$$\begin{aligned}
\phi_{k+1}(x) &\leq \frac{1}{k!}\int_{-\infty}^{x} (x-y)\, M(y)^k dm(y) \\
&= \frac{1}{k!}\int_{-\infty}^{x} (M(y) - k\,(x-y)\, M'(y))\, M(y)^{k-1} M'(y) dy \\
&\leq \frac{1}{k!}\int_{-\infty}^{x} M'(y) M(y)^k dy = \frac{M(x)^{k+1}}{(k+1)!},
\end{aligned}$$

which proves (2.9) for general k. Then the estimate of φ_λ is clear. □

Here we clarify the convergence of a sequence of monotone functions taking value ∞. For non-negative and non-decreasing function m which may take ∞, set

$$\widehat{m}(x) = \frac{2}{\pi} \tan^{-1} m(x), \qquad x \in \boldsymbol{R}.$$

Then

$$\widehat{m}(x) \in [0, 1]$$

and right continuous non-decreasing function satisfying

$$0 \le \widehat{m}(-\infty) \le \widehat{m}(x) \le \widehat{m}(l-) \le \widehat{m}(l) = 1,$$

if $l < \infty$. A sequence of non-negative and non-decreasing functions m_n is defined to converge to m as $n \to \infty$ if

$$\widehat{m}_n(x) \to \widehat{m}(x) \tag{2.10}$$

holds at any point of continuity of $\widehat{m}(x)$.

Lemma 2. *Suppose $m_n \in \mathcal{E}$ converges to $m \in \mathcal{E}$ as $n \to \infty$. Then it holds that*

$$\varliminf_{n\to\infty} l_n \ge l.$$

Proof. Let $x < l$ be a point of continuity for $\widehat{m}$. Then

$$\widehat{m}_n(x) \to \widehat{m}(x) < 1,$$

hence

$$\widehat{m}_n(x) < 1$$

for every sufficiently large n, which implies $x < l_n$ and completes the proof. □

The continuity of the correspondence from $\mathcal{E}$ to $\mathcal{S}$ is not hard to show. Let m_n, m be strings of $\mathcal{E}$ and define the convergence of m_n to m by

(A) $m_n(x) \to m(x)$ for every point of continuity of m.

(B) $\lim_{x\to-\infty} \sup_{n\ge1} M_n(x) = 0$

Theorem 2. *Suppose $m_n \in \mathcal{E}$ converge to $m \in \mathcal{E}$. Then, for every $\lambda < 0$ the Green functions $g_\lambda^{(n)}(x,y)$ of the string m_n converge to the Green function $g_\lambda(x,y)$ of m for any $x, y < l$. In particular the spectral functions $\sigma_n(\xi)$ converge to $\sigma(\xi)$ at every point of continuity of σ.*

Proof. Under the conditions it is easy to see that the $\varphi-$functions $\varphi_\lambda^{(n)}(x)$ of m_n converge to the $\varphi-$function $\varphi_\lambda(x)$ of m compact uniformly with respect to $(x,\lambda) \in (-\infty, l) \times \boldsymbol{C}$ from the uniform bound for $\varphi_\lambda^{(n)}$ due to Lemma 1.. Moreover, if $m(a) > 0$ at some a, a point of continuity of m, then there exists a positive constant C such that

$$\varphi_\lambda^{(n)}(y) \ge 1 - \lambda M_n(y) \ge 1 + C(y-a)$$

holds for any $y > a$, hence

$$f_\lambda^{(n)}(x) = \varphi_\lambda^{(n)}(x) \int_x^{l_n} \frac{1}{\varphi_\lambda^{(n)}(y)^2} dy$$

also converge to $f_\lambda(x)$. If supp$m = \phi$, namely $m(x) = 0$ identically on $(-\infty, l)$, then

$$f_\lambda^{(n)}(x) \to l - x$$

if $l < \infty$. The case $l = \infty$ is excluded. Consequently, we have

$$g_\lambda^{(n)}(x,y) = \varphi_\lambda^{(n)}(y) f_\lambda^{(n)}(x) \to \varphi_\lambda(y) f_\lambda(x) = g_\lambda(x,y)$$

for any $y \le x < l$. The identity

$$g_\lambda^{(n)}(x,y) = \int_0^\infty \frac{\varphi_\xi^{(n)}(x)\varphi_\xi^{(n)}(y)}{\xi - \lambda} d\sigma_n(\xi)$$

shows the last statement of the theorem. □

3. Scales and estimates by trace

The straight converse statement of the theorem 2 is hopeless to be true, because there is no characterization for a measure σ on $[0,\infty)$ to be a spectral measure of a string $m \in \mathcal{E}$. Therefore we prove the converse continuity of the correspondence by imposing a condition on $\{\sigma_n\}$. In the process of the proof we have to estimate $\varphi_\lambda(x)^{-2}$ in terms of m. A better way to investigate $\varphi_\lambda(x)^{-2}$ is to use probabilistic methods. Recall that for each fixed a, $\varphi_\lambda(a)$ has simple zeroes $\{\mu_n\}_{n\geq 1}$ which are eigenvalues of $-L$ on $(-\infty,a]$ with Dirichlet boundary condition at $x=a$. Since the Green function for this operator is

$$a-(x\vee y)\,,$$

we see

$$\sum_{n=1}^{\infty}\mu_n^{-1}=\mathrm{tr}\,(-L)^{-1}=\int_{-\infty}^{a}(a-x)\,dm(x)=M(a)<\infty. \tag{3.1}$$

Choosing a $b<l$, we denote by ϕ_λ^b $\left(\psi_\lambda^b\right)$ the solutions of

$$-\frac{d}{dm}\frac{d}{dx}f=\lambda f,\quad \text{with } f(b)=1,\ f'(b)=0\quad (f(b)=0,\ f'(b)=1)\ \text{respectively.}$$

Then we see an identity

$$\varphi_\lambda(x)=\varphi_\lambda(b)\phi_\lambda^b\,(x)+\varphi_\lambda'(b)\psi_\lambda^b\,(x)$$

holds. Lemma 1 implies $\varphi_\lambda(b)$ and $\varphi_\lambda'(b)$ are entire functions of at most exponential type $M(b)$ as functions of λ. On the other hand, $\phi_\lambda^b\,(x)$ and $\psi_\lambda^b\,(x)$ are entire functions of order at most $1/2$ as functions of λ. Therefore we know that $\varphi_\lambda(x)$ is an entire function of minimal exponential type, which combined with (3.1) shows

$$\varphi_\lambda(a)=\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\mu_n}\right).$$

For the detail refer to page 441 of [5]. Now let $\{X_n\}_{n\geq 1}$ be independent random variables each of which has an exponential distribution of mean 1. Then, an identity

$$E\exp\left(\lambda\sum_{n=1}^{\infty}\mu_n^{-1}X_n\right)=\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\mu_n}\right)^{-1}=\varphi_\lambda(a)^{-1}$$

holds. Therefore, letting $\left\{\widetilde{X}_n\right\}_{n\geq 1}$ be independent copies of $\{X_n\}_{n\geq 1}$ and setting

$$Y_n=X_n+\widetilde{X}_n,\quad X=\sum_{n=1}^{\infty}\mu_n^{-1}Y_n,$$

we have

$$\varphi_\lambda(a)^{-2}=E\exp\left(\lambda X\right). \tag{3.2}$$

We denote $X=X(a)$ if necessary, because the eigenvalues $\{\mu_n\}_{n\geq 1}$ depends on the boundary a.

Lemma 3. *Suppose the spectral measure σ of an $m\in\mathcal{E}$ satisfies*

$$p(t)=\int_0^\infty e^{-t\xi}d\sigma\,(\xi)<\infty\quad \textit{for any } t>0.$$

Then, for any non-negative Borel measurable function f on $[0,\infty)$

$$\int_{-\infty}^{l}Ef\left(X\left(x\right)\right)dx=\int_0^\infty p(t)f(t)dt \tag{3.3}$$

holds by permitting for the integrals to take the value ∞ simultaneously.

Proof. From (2.4) it follows that for any $x < l$ and $\lambda < 0$

$$\int_x^l Ee^{\lambda X(y)}dy = Ee^{\lambda X(x)}\int_0^\infty \frac{\varphi_\xi(x)^2}{\xi-\lambda}d\sigma(\xi) = \int_0^\infty Ee^{\lambda(X(x)+t)}p(t,x,x)dt,$$

holds, where $p(t,x,y)$ is the transition probability density defined by

$$p(t,x,y) = \int_0^\infty e^{-t\xi}\varphi_\xi(x)\varphi_\xi(y)\,d\sigma(\xi),$$

hence a functional monotone class theorem shows that the identity below holds for any non-negative bounded continuous function f on $[0,\infty)$.

$$\int_x^l E\left(f(X(y))\,e^{\lambda X(y)}\right)dy = \int_0^\infty E\left(f(X(x)+t)\,e^{\lambda(X(x)+t)}\right)p(t,x,x)dt \tag{3.4}$$

Since, for $t > 2M(x)$

$$p(t,x,x) = \int_0^\infty e^{-\xi t}\varphi_\xi(x)^2\,d\sigma(\xi) \le \int_0^\infty e^{-\xi t}e^{2\xi M(x)}d\sigma(\xi) = p(t-2M(x))$$

holds, assuming $f(t) = 0$ for $t < \epsilon$, we see

$$\int_{-\infty}^l E\left(f(X(y))\,e^{\lambda X(y)}\right)dy = \int_0^\infty f(t)\,e^{\lambda t}p(t)dt$$

by letting $x \to -\infty$. Here we have used the fact

$$X(x) \to 0 \quad \text{as } x \to -\infty.$$

The rest of the proof is a routine. □

Now we define a scale function ϕ on $[0,1]$ as a function satisfying the following properties.

(S.1) ϕ is strictly increasing, convex and $\phi(0) = 0$, $\phi'(1-) < \infty$.

(S.2) For each $x > 0$

$$\overline{\lim_{y\downarrow 0}}\frac{\phi(xy)}{\phi(y)} < \infty.$$

(S.3) For each $x \in (0,1]$

$$\underline{\lim_{y\downarrow 0}}\frac{\phi(xy)}{\phi(y)} > 0$$

The property (S.1) enables us to extend ϕ linearly to $[1,\infty)$, namely

$$\phi(x) = \phi(1) + \phi'(1-)(x-1)$$

for $x > 1$. Then ϕ becomes non-negative, convex and non-decreasing function on $[0,\infty)$. Throughout the paper ϕ is always extended to $[1,\infty)$ linearly in this way. A regularly varying function at 0 satisfies the condition (S.2), (S.3). Set

$$C_+(x) = \sup_{y>0}\frac{\phi(xy)}{\phi(y)} < \infty, \qquad C_-(x) = \inf_{y\in(0,1]}\frac{\phi(xy)}{\phi(y)} > 0.$$

Then C_+ becomes non-negative, convex and non-decreasing on $[0,\infty)$. It satisfies the submultiplicative property

$$C_+(xy) \le C_+(x)C_+(y)$$

for any $x, y > 0$, hence, setting

$$\alpha_+ = \sup_{x>1}\frac{\log C_+(e^x)}{x} \in [0,\infty)$$

we see

$$C_+(x) \le x^{\alpha_+}$$

holds for any $x \geq e$. α_+ should be not less than 1 due to the convexity of C_+. Since

$$\phi(xy) \leq C_+(x)\phi(y)$$

holds for any $x, y > 0$, we have

$$\phi(x) \geq \frac{\phi(1)}{C_+(1/x)} \geq \phi(1)\, x^{\alpha_+}$$

for any $x \in [0, 1/e]$. Therefore the property (S.2) restricts ϕ not to decay faster than with a power order.

$$\phi(xy) \leq C_+(x)\phi(y) \tag{3.5}$$

C_- satisfies

$$C_-(xy) \geq C_-(x)C_-(y)$$

for any $x, y \in [0, 1]$. Typical examples for functions satisfying (S.1)∼(S.3) are

$$\phi(x) = x^{\alpha}, \quad x^{\alpha}(c - \log x),$$

where $\alpha \geq 1$ and c is a sufficiently large positive constant.

Lemma 4. *Let $\{Y_n\}_{n\geq 1}$ be a sequence of identically distributed non-negative random variables with mean μ. and $\{\lambda_n\}_{n\geq 1}$ be a non-negative sequence satisfying*

$$\sum_{n=1}^{\infty} \lambda_n < \infty,$$

and set

$$X = \sum_{n=1}^{\infty} \lambda_n Y_n.$$

Then, we have
(1) *If ϕ satisfies (S.1), then*

$$\phi(EX) \leq E\phi(X).$$

(2) *If ϕ satisfies (S.1), (S.2), then*

$$E\phi(X) \leq \left(EC_+\left(\frac{Y_1}{\mu}\right)\right)\phi(EX).$$

Proof. Jensen's inequality implies the inequality in (1). To show the second inequality we set

$$m_n = m^{-1}\lambda_n, \quad m = \sum_{k=1}^{\infty} \lambda_k.$$

Then (3.5) implies

$$\phi(X) = \phi\left(m\mu \sum_{n=1}^{\infty} m_n \frac{Y_n}{\mu}\right) \leq \phi(m\mu)\, C_+\left(\sum_{n=1}^{\infty} m_n \frac{Y_n}{\mu}\right)$$

Since the function C_+ is convex, we have

$$C_+\left(\sum_{n=1}^{\infty} m_n \frac{Y_n}{\mu}\right) \leq \sum_{n=1}^{\infty} m_n C_+\left(\frac{Y_n}{\mu}\right),$$

and

$$E\phi(X) \leq \phi(m\mu) \sum_{n=1}^{\infty} m_n EC_+\left(\frac{Y_n}{\mu}\right) = \phi(EX)\, EC_+\left(\frac{Y_1}{\mu}\right).$$

□

Let X be the same as was defined in (3.2) and set

$$C_{\phi}=EC_{+}\left(\frac{Y_{1}}{\mu}\right)=\int_{0}^{\infty}te^{-t}C_{+}\left(t/2\right)dt<\infty.$$

Lemma 5. *We have*
(1) *If* ϕ *satisfies* $(S.1)$, *then*

$$E\left(\phi\left(X\right)e^{\lambda X}\right)\geq\varphi_{\lambda}\left(a\right)^{-2}\phi\left(\int_{-\infty}^{a}\varphi_{\lambda}\left(x\right)^{2}dm\left(x\right)\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy\right).$$

(2) *If* ϕ *satisfies* $(S.1),(S.2)$, *then*

$$E\left(\phi\left(X\right)e^{\lambda X}\right)\leq C_{\phi}\varphi_{\lambda}\left(a\right)^{-2}\phi\left(\int_{-\infty}^{a}\varphi_{\lambda}\left(x\right)^{2}dm\left(x\right)\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy\right).$$

Proof. For a fixed $\lambda<0$ let Z be a non-negative random variable satisfying

$$Ee^{\mu Z}=\varphi_{\lambda+\mu}\left(a\right)^{-2}\varphi_{\lambda}\left(a\right)^{2}=\prod_{n=1}^{\infty}\left(1-\frac{\mu}{\mu_{n}-\lambda}\right)^{-2}.$$

Then, note an identity

$$E\left(\phi\left(X\right)e^{\lambda X}\right)=\varphi_{\lambda}\left(a\right)^{-2}E\left(\phi\left(Z\right)\right), \tag{3.6}$$

which can be shown from the observation

$$\frac{\partial^{k}}{\partial\lambda^{k}}\varphi_{\lambda}\left(a\right)^{-2}=\varphi_{\lambda}\left(a\right)^{-2}\frac{\partial^{k}}{\partial\mu^{k}}\left(\varphi_{\lambda+\mu}\left(a\right)^{-2}\varphi_{\lambda}\left(a\right)^{2}\right)\Big|_{\mu=0}$$

for any $k\geq0$, because this implies the identity when $\phi\left(x\right)=x^{k}$. To apply Lemma 4 to Z we need to compute EZ. If we denote the Green operator for L on $(-\infty,a]$ with Dirichlet boundary condition at a by G_{λ}, then

$$G_{\lambda}\left(x,y\right)=\varphi_{\lambda}\left(y\right)\varphi_{\lambda}\left(x\right)\int_{x}^{a}\varphi_{\lambda}\left(z\right)^{-2}dz\quad\text{for }x\geq y$$

hence

$$EZ=\sum_{j=1}^{\infty}\frac{1}{\mu_{j}-\lambda}=\mathrm{tr}G_{\lambda}=\int_{-\infty}^{a}\varphi_{\lambda}\left(x\right)^{2}dm\left(x\right)\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy$$

holds, and we have the inequalities in the statement. □

The right side of the inequalities in Lemma 5 can be estimated further.

Lemma 6. *For* $\lambda<0$ *the following inequalities are valid.*
(1) $\displaystyle\int_{-\infty}^{a}\varphi_{\lambda}\left(x\right)^{2}dm\left(x\right)\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy\geq M(a)\varphi_{\lambda}\left(a\right)^{-2}$
(2) $\displaystyle\int_{-\infty}^{a}\varphi_{\lambda}\left(x\right)^{2}dm\left(x\right)\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy\leq M(a)\wedge\left(\frac{\log\varphi_{\lambda}\left(a\right)}{-\lambda}\right)$

Proof. The inequality (1) and the first inequality of (2) follow from the monotonicity of $\varphi_{\lambda}\left(z\right)$, namely we have

$$\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy\geq\varphi_{\lambda}\left(a\right)^{-2}\left(a-x\right),\quad\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy\leq\varphi_{\lambda}\left(x\right)^{-2}\left(a-x\right),$$

which implies

$$\int_{-\infty}^{a}\varphi_{\lambda}\left(x\right)^{2}dm\left(x\right)\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy\geq\varphi_{\lambda}\left(a\right)^{-2}\int_{-\infty}^{a}\left(a-x\right)dm\left(x\right)=\varphi_{\lambda}\left(a\right)^{-2}M(a),$$

and

$$\int_{-\infty}^{a}\varphi_{\lambda}\left(x\right)^{2}dm\left(x\right)\int_{x}^{a}\varphi_{\lambda}\left(y\right)^{-2}dy\leq\int_{-\infty}^{a}\left(a-x\right)dm(x)=M(a).$$

The second inequality of (2) follows by using the equation satisfied by $\varphi_\lambda(x)$

$$d\varphi'_\lambda(y) = -\lambda\varphi_\lambda(y)\, dm(y),$$

which yields

$$\begin{aligned}
&-\lambda \int_{-\infty}^{a} \varphi_\lambda(y)^2\, dm(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \\
&= \int_{-\infty}^{a} \varphi_\lambda(y)\, d\varphi'_\lambda(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \\
&= \varphi_\lambda(y)\varphi'_\lambda(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz\Big|_{-\infty}^{a} - \int_{-\infty}^{a} \varphi'_\lambda(y)^2\, dy \int_y^a \varphi_\lambda(z)^{-2}\, dz + \int_{-\infty}^{a} \frac{\varphi'_\lambda(y)}{\varphi_\lambda(y)} dy.
\end{aligned}$$

Noting

$$\varphi_\lambda(y)\varphi'_\lambda(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz \underset{y\to-\infty}{\sim} -\lambda m(y)(a-y) \underset{y\to-\infty}{\to} 0,$$

we see

$$\begin{aligned}
-\lambda \int_{-\infty}^{a} \varphi_\lambda(y)^2\, dm(y) \int_y^a \varphi_\lambda(z)^{-2}\, dz &= \log\varphi_\lambda(a) - \int_{-\infty}^{a} \varphi'_\lambda(y)^2\, dy \int_y^a \varphi_\lambda(z)^{-2}\, dz \\
&\le \log\varphi_\lambda(a),
\end{aligned}$$

which completes the proof. $\square$

As the last lemma in this section we have

Lemma 7. *The following two estimates hold.*
(1) *For any function ϕ satisfying $(S.1)$ it holds that*

$$\int_0^\infty p(t)\phi(t)\, e^{\lambda t} dt \ge \int_{-\infty}^{l} \phi\left(M(x)\varphi_\lambda(x)^{-2}\right) \varphi_\lambda(x)^{-2}\, dx$$

(2) *For any function ϕ satisfying $(S.1), (S.2)$ it holds that*

$$\begin{aligned}
\int_0^\infty p(t)\phi(t)\, e^{\lambda t} dt &\le C_\phi \int_{-\infty}^{l} \phi\left(M(x) \wedge \left(\frac{\log\varphi_\lambda(x)}{-\lambda}\right)\right) \varphi_\lambda(x)^{-2}\, dx \\
&\le C_\phi \int_{-\infty}^{a} \phi(M(x))\, \varphi_\lambda(x)^{-2}\, dx + C_\phi \frac{-\lambda}{\varphi'_\lambda(a)} \int_0^\infty \phi(t)\, e^{\lambda t} dt,
\end{aligned}$$

Proof. All we have to show is an estimate of the integral

$$\int_a^l \phi\left(\frac{\log\varphi_\lambda(x)}{-\lambda}\right) \varphi_\lambda(x)^{-2}\, dx.$$

Noting the monotonicity of $\varphi_\lambda(x), \varphi'_\lambda(x)$ and $\varphi_\lambda(x) \ge 1$, we see

$$\begin{aligned}
&\int_a^l \phi\left(\frac{\log\varphi_\lambda(x)}{-\lambda}\right) \varphi_\lambda(x)^{-2}\, dx \\
&= \int_{\varphi_\lambda(a)}^{\varphi_\lambda(l)} \phi\left(\frac{\log z}{-\lambda}\right) \frac{1}{z^2\varphi'_\lambda\left(\varphi_\lambda^{-1}(z)\right)} dz \\
&\le \frac{1}{\varphi'_\lambda\left(\varphi_\lambda^{-1}(\varphi_\lambda(a))\right)} \int_{\varphi_\lambda(a)}^{\varphi_\lambda(l)} \phi\left(\frac{\log z}{-\lambda}\right) \frac{dz}{z^2} \le \frac{-\lambda}{\varphi'_\lambda(a)} \int_0^\infty \phi(t)\, e^{\lambda t} dt.
\end{aligned}$$

$\square$

4. Continuity of the correspondence from $\mathcal{S}$ to $\mathcal{E}$

In this section we give a partial converse of Theorem 2. The lemma below will be useful later.

Lemma 8. *Let $m_n \in \mathcal{E}$ and σ_n be its spectral function. Suppose*

$$\lim_{n\to\infty} M_n(l_n) = 0 \tag{4.1}$$

holds. Then $\sigma_n(\xi) \to 0$ for any $\xi > 0$.

Proof. Since $M_n(l_n) < \infty$, we have $l_n < \infty$. Set $\widetilde{m}_n(x) = m_n(x + l_n)$. Then

$$\widetilde{M}_n(0) \to 0 \tag{4.2}$$

and its spectral measure coincides with σ_n. Since the condition (4.2) implies

$$\widetilde{m}_n(x) \to \begin{cases} 0 & \text{for} \quad x < 0 \\ \infty & \text{for} \quad x > 0 \end{cases}$$

in $\mathcal{E}$, Theorem 2 shows $\sigma_n \to 0$. □

Theorem 3. *Let $m_n \in \mathcal{E}$ and σ_n be its spectral function satisfying*

$$p_n(t) = \int_0^\infty e^{-t\xi} d\sigma_n(\xi) < \infty \quad \textit{for any } t > 0$$

and

$$\sup_{n\geq 1} \int_0^1 p_n(t)\phi(t)dt < \infty \tag{4.3}$$

for a function ϕ satisfying (S.1). Assume there exists a non-trivial measure σ on $[0,\infty)$ satisfying

$$\sigma_n(\xi) \to \sigma(\xi)$$

at every point of continuity of σ. Then

$$\lim_{n\to\infty} p_n(t) = p(t), \qquad \varliminf_{n\to\infty} M_n(l_n) > 0$$

hold. Choose c such that

$$0 < c < \varliminf_{n\to\infty} M_n(l_n)$$

and define a_n by the solution $M_n(a_n) = c$. Then there exists a unique $m \in \mathcal{E}^{(c)}$ with spectral measure σ and it holds that $m_n(\cdot + a_n) \to m$ in $\mathcal{E}$, hence $\sigma \in \mathcal{S}$.

Proof. For any $\epsilon > 0$ and any $N > 0$

$$\int_0^\epsilon p_n(t)\phi(t)dt = \int_0^\epsilon \phi(t)dt \int_0^\infty e^{-t\xi} d\sigma_n(\xi) \geq e^{\epsilon N} \int_N^\infty e^{-2\epsilon\xi} d\sigma_n(\xi) \int_0^\epsilon \phi(t)dt$$

holds, and the condition (4.3) implies that there exists a constant C_ϵ such that

$$\int_N^\infty e^{-2\epsilon\xi} d\sigma_n(\xi) \leq e^{-\epsilon N} \frac{\int_0^\epsilon p_n(t)\phi(t)dt}{\int_0^\epsilon \phi(t)dt} \leq e^{-\epsilon N} C_\epsilon$$

is valid for any n, N, which yields

$$p_n(t) \to p(t) = \int_0^\infty e^{-t\xi} d\sigma(\xi) \tag{4.4}$$

as $n \to \infty$. Applying (3.3) to $\phi(t)e^{\lambda t}$ for $\lambda < 0$ shows

$$\begin{aligned} \int_{-\infty}^{l_n} E\left(\phi(X_n(x))\, e^{\lambda X_n(x)}\right) dx &= \int_0^\infty p_n(t) e^{\lambda t}\phi(t)dt \\ &\leq \int_0^1 p_n(t)e^{\lambda t}\phi(t)dt + p_n(1)\int_1^\infty e^{\lambda t}\phi(t)dt \leq C \end{aligned}$$

with a constant C, where $X_n(x)$ is defined by the eigenvalues $\{\mu_j(x)\}_{j\geq 1}$ corresponding to m_n. Since we assume the limiting spectral measure σ is non-trivial, Lemma 8 shows

$$\varliminf_{n\to\infty} M_n(l_n) > 0.$$

For c such that

$$0 < c < \varliminf_{n\to\infty} M_n(l_n)$$

define a_n by the solution $M_n(a_n) = c$ and set

$$\widetilde{m}_n(x) = m_n(x + a_n).$$

Then $\widetilde{m}_n \in \mathcal{E}^{(c)}$ and an inequality (1) of Lemma 7 implies

$$\int_{-\infty}^{l_n} E\left(\phi(X_n(x))\, e^{\lambda X_n(x)}\right) dx \geq \int_{-\infty}^{\widetilde{l}_n} \widetilde{\varphi}_\lambda^{(n)}(x)^{-2}\, \phi\left(\widetilde{M}_n(x)\, \widetilde{\varphi}_\lambda^{(n)}(x)^{-2}\right) dx,$$

and from Lemma 1 we have

$$\begin{aligned}\int_{-\infty}^{\widetilde{l}_n} \widetilde{\varphi}_\lambda^{(n)}(x)^{-2}\, \phi\left(\widetilde{M}_n(x)\, \widetilde{\varphi}_\lambda^{(n)}(x)^{-2}\right) dx &\geq \int_{-\infty}^{l_n} e^{2\lambda \widetilde{M}_n(x)} \phi\left(\widetilde{M}_n(x)\, e^{2\lambda \widetilde{M}_n(x)}\right) dx \\ &\geq \int_{-\infty}^{0} e^{2\lambda c} \phi\left(\widetilde{M}_n(x)\, e^{2\lambda c}\right) dx.\end{aligned}$$

Thus

$$\int_{-\infty}^{0} e^{2\lambda c} \phi\left(\widetilde{M}_n(x)\, e^{2\lambda c}\right) dx \leq C$$

is valid for any $n \geq 1$. Therefore, for any $x < 0$

$$e^{2\lambda c}(-x)\, \phi\left(\widetilde{M}_n(x)\, e^{2\lambda c}\right) \leq \int_{x}^{0} e^{2\lambda c} \phi\left(\widetilde{M}_n(y)\, e^{2\lambda c}\right) dy \leq C$$

which implies that $\{\widetilde{m}_n\}_{n\geq 1}$ has a convergent subsequence in the sense of the convergence in $\mathcal{E}$, namely the convergence under the conditions (A), (B). Since we have proved (4.4), the uniqueness of the spectral measure in $\mathcal{E}^{(c)}$ and Theorem 2 complete the proof. $\square$

Corollary 1. *Suppose a measure σ on $[0, \infty)$ satisfies*

$$\int_0^1 p(t)\phi(t)dt < \infty$$

with a function ϕ on $[0, 1]$ satisfying (S.1). Then, there exists an $m \in \mathcal{E}$ with spectral measure σ in $\mathcal{S}$.

Proof. Define σ_n by

$$\sigma_n(\xi) = \begin{cases} \sigma(\xi) & \text{for} \quad \xi < n \\ \sigma(n) & \text{for} \quad \xi \geq n \end{cases}.$$

Then this σ_n satisfies all the conditions of Theorem 3. Applying the theorem we easily obtain the corollary. $\square$

This corollary provides a plenty of spectral measures in $\mathcal{S}$ growing faster than any power order at ∞.

5. Continuity of the correspondence between $\mathcal{E}_\phi$ and $\mathcal{S}_\phi$

In this section we give a necessary and sufficient condition for the continuity of the correspondence by restricting the order of growth of spectral measures at ∞. We call ϕ to be a scale function if it satisfies the conditions (S.1), (S.2), (S.3). For a scale function ϕ set

$$\mathcal{E}_\phi = \left\{ m \in \mathcal{E};\ \int_{-\infty}^{a} \phi\left(M\left(x\right)\right) dx < \infty \ \text{ for } \exists a \in \left(l_-, l_+\right) \right\},$$

and

$$\mathcal{S}_\phi = \left\{ \sigma;\ \int_1^\infty \widetilde{\phi}\left(\xi\right) \sigma\left(d\xi\right) < \infty \right\},$$

where

$$\widetilde{\phi}\left(\xi\right) = \int_0^\infty e^{-t\xi} \phi\left(t\right) dt.$$

It is easy to see that

$$\int_1^\infty \widetilde{\phi}\left(\xi\right) \sigma\left(d\xi\right) < \infty \Longleftrightarrow \int_0^1 p(t)\phi\left(t\right) dt < \infty.$$

Moreover, from the properties of scales, it is always valid that for $\sigma \in \mathcal{S}_\phi$

$$\int_1^\infty \xi^{-\alpha-1} \sigma\left(d\xi\right) < \infty$$

for an $\alpha \geq 1$. On the other hand, if $m \in \mathcal{E}_\phi$, (2) of Lemma 7 yields

$$\begin{aligned}\int_0^\infty p(t)\phi\left(t\right) e^{\lambda t} dt &\leq C_\phi \int_{-\infty}^{a} \phi\left(M(x)\right) \varphi_\lambda\left(x\right)^{-2} dx + C_\phi \frac{-\lambda}{\varphi_\lambda'\left(a\right)} \int_0^\infty \phi\left(t\right) e^{\lambda t} dt \\ &\leq C_\phi \int_{-\infty}^{a} \phi\left(M(x)\right) dx + C_\phi \frac{-\lambda}{\varphi_\lambda'\left(a\right)} \int_0^\infty \phi\left(t\right) e^{\lambda t} dt.\end{aligned}$$

Therefore it holds that

$$\int_0^\infty p(t)\phi\left(t\right) e^{\lambda t} dt < \infty,$$

which shows $\sigma \in \mathcal{S}_\phi$. Conversely assume $\sigma \in \mathcal{S}_\phi$. Then

$$\begin{aligned}\int_0^\infty p(t)\phi\left(t\right) e^{\lambda t} dt &= \int_0^1 p(t)\phi\left(t\right) e^{\lambda t} dt + \int_1^\infty p(t)\phi\left(t\right) e^{\lambda t} dt \\ &\leq \int_0^1 p(t)\phi\left(t\right) dt + p(1)\int_1^\infty \phi\left(t\right) e^{\lambda t} dt < \infty\end{aligned}$$

for any $\lambda < 0$. Therefore

$$\int_{-\infty}^{l} \phi\left(M(x)\varphi_\lambda\left(x\right)^{-2}\right) \varphi_\lambda\left(x\right)^{-2} dx < \infty$$

holds. Since ϕ satisfies the condition (S.3)

$$\phi\left(M(x)\varphi_\lambda\left(x\right)^{-2}\right) \geq C_-\left(\varphi_\lambda\left(x\right)^{-2}\right) \phi\left(M(x)\right)$$

holds. Due to

$$\varphi_\lambda\left(x\right)^{-2} \geq e^{2\lambda M(x)}$$

and $M(x) \to 0$ as $x \to -\infty$, we easily see

$$\int_{-\infty}^{a} \phi\left(M(x)\right) dx < \infty,$$

which implies $m \in \mathcal{E}_\phi$. Therefore, a string belongs to $\mathcal{E}_\phi$ if and only if its spectral measure is an element of $\mathcal{S}_\phi$.

For ϕ let m_n, m be strings of $\mathcal{E}_\phi$ and define the convergence of m_n to m in $\mathcal{E}_\phi$ by

(C) $\lim_{x\to-\infty}\sup_{n\geq 1}\int_{-\infty}^{x}\phi\left(M_n(y)\right)dy = 0,$

in addition to the condition (A). The convergence of spectral measures in $\mathcal{S}_\phi$ is defined by

(A') $\sigma_n(\xi)\to\sigma(\xi)$ at every point of continuity of σ.

(C') $\lim_{N\to\infty}\sup_{n\geq 1}\int_{N}^{\infty}\widetilde{\phi}(\xi)\,\sigma_n(d\xi) = 0.$

An equivalent statement is possible by $p(t)$.

(A'') $p_n(t)\to p(t)$ for any $t>0$.

(C'') $\lim_{\epsilon\downarrow 0}\sup_{n\geq 1}\int_{0}^{\epsilon}p_n(t)\,\phi(t)\,dt = 0.$

Set

$$\mathcal{E}_\phi^{(c)} = \mathcal{E}_\phi\cap\mathcal{E}^{(c)}.$$

Theorem 4. *Let $\{\sigma_n\}_{n\geq 1},\sigma$ be elements of $\mathcal{S}_\phi$ and $m_n, m\in\mathcal{E}_\phi^{(c)}$ be the strings corresponding to σ_n, σ respectively. Then, $m_n\to m$ in $\mathcal{E}_\phi^{(c)}$ if and only if $\sigma_n\to\sigma$ in $\mathcal{S}_\phi$.*

Proof. Suppose $m_n\to m$ in $\mathcal{E}_\phi^{(c)}$. Then, Theorem 2 shows the validity of the condition (A'). Therefore we have only to check the condition (C''). From (2) of Lemma 7

$$\int_0^\infty p_n(t)\phi(t)\,e^{\lambda t}dt \leq C_\phi\int_{-\infty}^{0}\phi\left(M_n(x)\right)\varphi_\lambda^{(n)}(x)^{-2}\,dx + C_\phi\frac{-\lambda}{\varphi_\lambda^{(n)\prime}(0)}\int_0^\infty\phi(t)\,e^{\lambda t}dt$$

is valid. Fix $\epsilon>0$ and choose $a<0$ such that

$$C_\phi\int_{-\infty}^{a}\phi\left(M_n(x)\right)dx < \epsilon$$

for any $n\geq 1$. Since $M_n(x),\varphi_\lambda^{(n)}(x)$ converge to $M(x),\varphi_\lambda(x)$ uniformly on $(-\infty,0]$ and the estimate

$$\varphi_\lambda^{(n)}(0)\geq 1-\lambda M_n(0) = 1-\lambda c$$

show that if $-\lambda$ is sufficiently large, then

$$C_\phi\int_{a}^{0}\phi\left(M_n(x)\right)\varphi_\lambda^{(n)}(x)^{-2}\,dx < \epsilon$$

is valid for any $n\geq 1$. Moreover, due to $c>0$

$$\underline{\lim}_{n\to\infty}\,m_n(0)\geq m(0-)>0$$

holds, hence

$$C_\phi\frac{-\lambda}{\varphi_\lambda^{(n)\prime}(0)}\int_0^\infty\phi(t)\,e^{\lambda t}dt\leq C_\phi\frac{1}{m_n(0)}\int_0^\infty\phi(t)\,e^{\lambda t}dt<\epsilon$$

also holds for any $n\geq 1$ if we choose sufficiently large $-\lambda$, which implies

$$\sup_{n\geq 1}\int_0^\infty p_n(t)\phi(t)\,e^{\lambda t}dt\leq 3\epsilon.$$

From

$$\int_0^{-1/\lambda}p_n(t)\phi(t)\,dt\leq e\int_0^\infty p_n(t)\phi(t)\,e^{\lambda t}dt\leq 3e\epsilon$$

the condition (C'') is confirmed. Conversely assume $\sigma_n \to \sigma$ in $\mathcal{S}_\phi$. Then Theorem 3 shows the condition (A) holds. Hence we have only to check the condition (C). From (1) of Lemma 7

$$\int_0^\infty p_n(t)\phi(t)\, e^{\lambda t}dt \geq \int_{-\infty}^l \phi\left(M_n(x)\varphi_\lambda^{(n)}(x)^{-2}\right)\varphi_\lambda^{(n)}(x)^{-2}\, dx$$

follows. The property (S.3) implies

$$\phi\left(M_n(x)\varphi_\lambda^{(n)}(x)^{-2}\right) \geq C_-\left(\varphi_\lambda^{(n)}(x)^{-2}\right)\phi\left(M_n(x)\right)$$

and, as was pointed out in the proof of Theorem 3, $\varphi_\lambda^{(n)}(x) \to 1$ as $x \to -\infty$ uniformly with respect to n. Therefore, the condition (C'') guarantees the condition (C). □

The last theorem can be restated as the convergence in $\mathcal{E}_\phi$.

Theorem 5. *Let $\{\sigma_n\}_{n\geq 1}, \sigma$ be elements of $\mathcal{S}_\phi$ and $m_n, m \in \mathcal{E}_\phi$ be the strings corresponding to σ_n, σ respectively. Assume $\sigma_n \to \sigma$ in $\mathcal{S}_\phi$ and σ is non-trivial. Then, there exist a sequence $\{a_n\}_{n\geq 1}$ in $\boldsymbol{R}$ and $c > 0$ with $m_n(\cdot + a_n) \in \mathcal{E}_\phi^{(c)}$, and*

$$m_n(\cdot + a_n) \to m$$

holds in $\mathcal{E}_\phi$.

Proof. Since σ is non-trivial, we can apply Lemma 8. The rest of the proof is clear from Theorem 4. □

For applications it will be helpful to rewrite the condition (C) as Kasahara-Watanabe did in [4].

Lemma 9. *Assume ϕ satisfies (S.1). Then a sequence $\{m_n\}_{n\geq 1}$ converges to m in $\mathcal{E}_\phi$, if and only if $\{m_n\}_{n\geq 1}$ and m satisfy the condition below.*

(D) For any $x \in \boldsymbol{R}$

$$\int_{-\infty}^x \phi\left(M_n(y)\right)dy \to \int_{-\infty}^x \phi\left(M(y)\right)dy.$$

Similarly the set of conditions (A') and (C') is equivalent to (D'), and that of (A'') and (C'') is equivalent to (D'').

(D') For any $\lambda < 0$

$$\int_0^\infty \widetilde{\phi}\left(\xi - \lambda\right)\sigma_n\left(d\xi\right) \to \int_0^\infty \widetilde{\phi}\left(\xi - \lambda\right)\sigma\left(d\xi\right).$$

(D'') For any $\lambda < 0$

$$\int_0^\infty p_n(t)\,\phi(t)\,e^{t\lambda}dt \to \int_0^\infty p(t)\,\phi(t)\,e^{t\lambda}dt.$$

Proof. Assume $m_n \to m$ in $\mathcal{E}_\phi$. Then, (C) implies that there exists $c < l$ such that

$$\int_{-\infty}^c \phi\left(M_n(y)\right)dy \leq 1,$$

hence for any $x < c$

$$\phi\left(M_n(x)\right)(c - x) \leq \int_x^c \phi\left(M_n(y)\right)dy \leq \int_{-\infty}^c \phi\left(M_n(y)\right)dy \leq 1,$$

which shows

$$M_n(x) \leq \phi^{-1}\left(\frac{1}{c-x}\right)$$

for any $n \geq 1$ and $x < c$. Then it is easy to see that $M_n(x) \to M(x)$ at every point x, and this together with (C) implies (D). Conversely, for any $\epsilon > 0$, choose $c < l$ such that

$$\int_{-\infty}^{c} \phi\left(M(y)\right) dy < \epsilon.$$

Then, clearly (C) follows from (D). The condition (A) can be derived from (D) by the monotonicity of ϕ and M_n. We omit the proof for (D') and (D''). $\square$

6. Application

Typical examples of m belonging to $\mathcal{E}$ are

$$m_\alpha(x) = \begin{cases} C_\alpha x^{-\beta} & x > 0 \quad \text{if } 0 < \alpha < 1 \\ e^x & x \in \mathbf{R} \quad \text{if } \alpha = 1 \\ C_\alpha \left(-x\right)^{-\beta} & x < 0 \quad \text{if } \alpha > 1. \end{cases} \qquad \text{with } \beta = \frac{\alpha}{\alpha - 1}$$

and the spectral measures and $p(t)$ are

$$\sigma_\alpha(d\xi) = \frac{\alpha^{2\alpha}}{\Gamma\left(1+\alpha\right)^2} d\xi^\alpha, \quad p_\alpha\left(t\right) = \frac{\alpha^{2\alpha}}{\Gamma\left(1+\alpha\right)} t^{-\alpha},$$

where

$$C_\alpha = \begin{cases} \left(\dfrac{1-\alpha}{\alpha}\right)^{\frac{\alpha}{1-\alpha}} & , \quad 0 < \alpha < 1 \\ \left(\dfrac{\alpha - 1}{\alpha}\right)^{-\frac{\alpha}{\alpha-1}} & , \quad 1 < \alpha \end{cases}.$$

In this section we consider the asymptotic behavior of the spectral measures and the transition probability densities when strings are close to the above typical ones. If $\alpha \in (0, 2)$, the following results are already known. Here we denote

$$f\left(x\right) \sim g\left(x\right) \quad \text{as} \quad x \uparrow 0, \left(x \to \infty\right)$$

if

$$\lim_{x \uparrow 0} \frac{f\left(x\right)}{g\left(x\right)} = 1, \quad \left(\lim_{x \to \infty} \frac{f\left(x\right)}{g\left(x\right)} = 1\right)$$

hold respectively. Let φ be a function regularly varying at 0 with exponent $\alpha - 1$.

Theorem 6. (*Kasahara*[1], *Kasahara-Watanabe*[4]) *The following asymptotic relationship between m and p is valid.*
(1) *If $\alpha \in (0,1)$, then*

$$m(x) \sim \frac{\left(-\beta\right)^\beta}{x\varphi^{-1}\left(x\right)} \quad \text{as } x \uparrow \infty$$

holds if and only if

$$p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma\left(1+\alpha\right)} \frac{1}{t} \varphi\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty$$

(2) *If $\alpha \in (1, 2)$, then*

$$m(x) \sim \frac{\beta^\beta}{-x\varphi^{-1}\left(-x\right)} \quad \text{as } x \uparrow 0$$

holds if and only if

$$p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma\left(1+\alpha\right)} \frac{1}{t} \varphi\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty$$

They showed an analogous result in case $\alpha = 1$ in Kasahara-Watanabe[4]. In this section we extend their results to the case $\alpha \geq 2$ by applying Theorem 4. The basic idea, which was first employed by Kasahara[1], is to use the continuity between m and p and the scaling relationship

$$abm(ax) \leftrightarrow \frac{1}{ab}p\left(b^{-1}t\right) \tag{6.1}$$

for any $a, b > 0$. The proof proceeds just like Kasahara-Watanabe[4], especially the case $\alpha = 1$.

Let $m \in \mathcal{E}$ be a non-decreasing function with $l = 0$, namely

$$m(x) < \infty \ \text{ on } \ (-\infty, 0) \quad \text{and } m(x) = \infty \ \text{ on } \ (0, \infty).$$

Let φ be a regularly varying function at 0 with exponent $\alpha - 1$ and set

$$m_\nu(x) = \nu\varphi^{-1}(\nu)\, m(\nu x).$$

Then from (6.1) we have

$$\begin{cases} M_\nu(x) = \varphi^{-1}(\nu)\, M(\nu x) \\ p_\nu(t) = \nu^{-1}\varphi^{-1}(\nu)^{-1}\, p\left(\varphi^{-1}(\nu)^{-1}\, t\right) \\ \sigma_\nu(\xi) = \nu^{-1}\varphi^{-1}(\nu)^{-1}\, \sigma\left(\varphi^{-1}(\nu)\,\xi\right) \end{cases}. \tag{6.2}$$

To consider an extension of Theorem 6 we introduce conditions on m and σ ;

$$m(x) \sim \frac{\beta^\beta}{-x\varphi^{-1}(-x)} \quad \text{as} \ \ x \uparrow 0, \tag{6.3}$$

which means

$$M(x) \sim \frac{\beta^\beta}{(\beta - 1)\,\varphi^{-1}(-x)} \quad \text{as} \ \ x \uparrow 0, \tag{6.4}$$

and

$$p(t) = \int_0^\infty e^{-t\xi} d\sigma(\xi) < \infty \quad \text{for any } t > 0. \tag{6.5}$$

Proposition 1. *If $m \in \mathcal{E}$ satisfies* (6.3), *then it holds that*

$$\sigma(\xi) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^\alpha \quad \textit{as} \ \ \xi \downarrow 0. \tag{6.6}$$

Moreover, if m satisfies (6.5) *as well, then* (6.7) *below holds.*

$$p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)}\frac{1}{t}\varphi\left(\frac{1}{t}\right) \quad \textit{as} \ \ t \to \infty. \tag{6.7}$$

Proof. Since

$$M_\nu(x) = \int_{-\infty}^x m_\nu(y)\, dy = \varphi^{-1}(\nu)\, M(\nu x)$$

holds, from (6.4) we know

$$M_\nu(x) \to \frac{\beta^\beta}{\beta - 1}(-x)^{1-\beta} \quad \text{as } \nu \to 0$$

for any $x < 0$, which means that $\{M_\nu\}$ satisfies the condition (B). Applying Theorem 2 yields

$$\sigma_\nu(\xi) \to \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^\alpha \quad \text{for any } \xi > 0$$

as $\nu \to 0$, which is equivalent to (6.6) due to (6.2). If we assume the condition (6.5) as well on σ, the Abelian theorem for Laplace transform shows the property (6.7). $\square$

To obtain a converse statement to the above proposition we need

Lemma 10. *Assume $\sigma \in \mathcal{S}$ satisfies the condition* (6.5) *and a condition*

$$\int_0^1 p(t)\phi(t)\,dt < \infty \tag{6.8}$$

for a positive function ϕ on $[0,1]$ satisfying

$$\phi(st) \le Ct^k\phi(s) \quad \text{for any } s,t \le 1 \quad \text{for some } k > \alpha - 1. \tag{6.9}$$

Then $\{p_\nu(t)\}$ satisfies the condition (28), *namely*

$$\sup_{\nu>0} \int_0^1 p_\nu(t)\,\phi(t)\,dt < \infty. \tag{6.10}$$

Proof. Since φ is a regularly varying function at 0 with exponent $\alpha - 1$, $t^{-1}\varphi\left(t^{-1}\right)$ is regularly varying at ∞ with exponent $-\alpha$, and there exists a slowly varying function $l(t)$ such that

$$\frac{1}{t}\varphi\left(\frac{1}{t}\right) = t^{-\alpha}l(t).$$

Generally a slowly varying function $l(t)$ has an expression

$$l(t) = c(t)\exp\left(\int_a^t \frac{\epsilon(u)}{u}du\right) \tag{6.11}$$

with a positive constant a and functions $c(t)$, $\epsilon(t)$ behaving as

$$c(t) \to c > 0, \quad \epsilon(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Now we decompose the integral in (6.10) into two parts:

$$\int_0^1 p_\nu(t)\,\phi(t)\,dt = \nu^{-1}\varphi^{-1}(\nu)^{-1}\int_0^1 p\left(\varphi^{-1}(\nu)^{-1}t\right)\phi(t)\,dt = I_1 + I_2$$

with

$$\begin{cases} I_1 = \nu^{-1}\varphi^{-1}(\nu)^{-1}\int_{N\varphi^{-1}(\nu)}^1 p\left(\varphi^{-1}(\nu)^{-1}t\right)\phi(t)\,dt, \\ I_2 = \nu^{-1}\varphi^{-1}(\nu)^{-1}\int_0^{N\varphi^{-1}(\nu)} p\left(\varphi^{-1}(\nu)^{-1}t\right)\phi(t)\,dt, \end{cases}$$

where N is chosen so that

$$|\epsilon(u)| \le \delta \quad \text{for any } u \ge N$$

holds with a positive δ satisfying $\delta < k - (\alpha - 1)$. Since the condition (6.7) implies

$$0 < \frac{p\left(\varphi^{-1}(\nu)^{-1}t\right)}{\varphi^{-1}(\nu)^{\alpha}t^{-\alpha}l\left(\varphi^{-1}(\nu)^{-1}t\right)} \le C'$$

for any $t \ge N\varphi^{-1}(\nu)$ with some constant C', we have

$$I_1 \le C'\nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha}\,l\left(\varphi^{-1}(\nu)^{-1}\right)\int_{N\varphi^{-1}(\nu)}^1 t^{-\alpha}\frac{l\left(\varphi^{-1}(\nu)^{-1}t\right)}{l\left(\varphi^{-1}(\nu)^{-1}\right)}\phi(t)\,dt.$$

First note

$$\nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha}\,l\left(\varphi^{-1}(\nu)^{-1}\right) = \nu^{-1}\varphi^{-1}(\nu)^{-1+\alpha}\,\varphi^{-1}(\nu)^{1-\alpha}\,\varphi\left(\varphi^{-1}(\nu)\right) = 1,$$

and the (6.11) shows for $t \geq N\varphi^{-1}(\nu)$

$$\begin{aligned}\frac{l\left(\varphi^{-1}(\nu)^{-1}t\right)}{l\left(\varphi^{-1}(\nu)^{-1}\right)} &= \exp\left(\int_a^{\varphi^{-1}(\nu)^{-1}t}\frac{\epsilon(u)}{u}du - \int_a^{\varphi^{-1}(\nu)^{-1}}\frac{\epsilon(u)}{u}du\right) \\ &= \exp\left(-\int_{\varphi^{-1}(\nu)^{-1}t}^{\varphi^{-1}(\nu)^{-1}}\frac{\epsilon(u)}{u}du\right) \\ &\leq \exp\left(\delta\log t^{-1}\right) = t^{-\delta}.\end{aligned}$$

In (6.9) setting $s = 1$, we have $\phi(t) \leq C\phi(1)t^k$, hence

$$I_1 \leq C\int_{N\varphi^{-1}(\nu)}^1 t^{-\alpha}t^{-\delta}\phi(t)\,dt \leq CC'\phi(1)\int_0^1 t^{-\alpha-\delta+k}dt$$

is valid. Due to (6.9) I_2 can be estimated as

$$\begin{aligned}I_2 &= \nu^{-1}\int_0^N p(s)\,\phi\left(\varphi^{-1}(\nu)\,s\right)ds \\ &\leq C\nu^{-1}\left(\varphi^{-1}(\nu)\right)^k\int_0^N p(s)\,\phi(s)\,ds \leq C''\nu^{-1+\frac{k}{\alpha-1}-\delta'}\int_0^N p(s)\,\phi(s)\,ds,\end{aligned}$$

where $\delta' > 0$ can be chosen so that

$$-1 + \frac{k}{\alpha-1} - \delta' > 0$$

holds. Consequently we have

$$\int_0^1 p_\nu(t)\,\phi(t)\,dt \leq CC'\phi(1)\int_0^1 t^{-\alpha-\delta+k}dt + C''\nu^{-1+\frac{k}{\alpha-1}-\delta'}\int_0^N p(s)\,\phi(s)\,ds,$$

and (6.8) implies the second assertion of (6.10). □

Remark 1. *The property* (6.9) *are satisfied not only by* $\phi(t) = t^k$ *with* $k > \alpha - 1$ *but also by subexponential functions: for* $p > 1, c > 0$

$$\phi(t) = \exp\left(-c\left(-\log t\right)^p\right).$$

Within the knowledge of the previous sections the best converse statement to Proposition 1 is as follows.

Proposition 2. *Let* $m \in \mathcal{E}$ *be a non-decreasing function with* $l = 0$ *and* $M(0) = \infty$. *Assume* $\sigma \in \mathcal{S}$ *satisfies the conditions* (6.5) *and* (6.8) *with a positive function* ϕ *on* $[0, 1]$ *satisfying* $(S.1)$ *and* (6.9). *Then the property* (6.6) (*equivalently* (6.7)) *implies* (6.3).

Proof. First note (6.6) is equivalent to

$$\sigma_\nu(\xi) \to \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)^2}\xi^\alpha \quad \text{for any } \xi > 0,$$

as $\nu \to 0$. Since we are assuming $M(0) = \infty$,

$$M_\nu(0) = \varphi^{-1}(\nu)\,M(0) = \infty$$

holds for any $\nu > 0$, and there exists uniquely $a_\nu < 0$ such that

$$M_\nu(a_\nu) = \frac{\beta^\beta}{\beta-1} \equiv c.$$

Set

$$\widetilde{M}_\nu(x) = M_\nu(x + a_\nu + 1).$$

Then, taking -1 instead of 0 as a normalization point, Lemma 10 makes it possible to apply Theorem 3 and we have

$$\widetilde{M}_{\nu}(x) \to \begin{cases} c(-x)^{1-\beta} & \text{for } x<0 \\ \infty & \text{for } x>0 \end{cases}$$

holds in $\mathcal{E}$ as $\nu \to 0$, from which

$$\varphi^{-1}(\nu) M(\nu(x+a_{\nu}+1)) \to \begin{cases} c(-x)^{1-\beta} & \text{for } x<0 \\ \infty & \text{for } x>0 \end{cases} \tag{6.12}$$

follows. To simplify the involved formula (6.12) we take their inverse. Set

$$u = M(\nu(x+a_{\nu}+1)), \quad \lambda = c\varphi^{-1}(\nu)^{-1}.$$

Since $\varphi^{-1}(\nu) M(\nu a_{\nu}) = c$, we easily see

$$\varphi\left(c\lambda^{-1}\right)(x+1) + M^{-1}(\lambda) = M^{-1}(u).$$

Denoting $y = \lambda^{-1}u$, (6.12) is equivalent to

$$y \to (-x)^{1-\beta},$$

from which

$$\frac{M^{-1}(\lambda y) - M^{-1}(\lambda)}{\varphi(c\lambda^{-1})} = x+1 \to 1 - y^{-(\beta-1)}$$

follows for any $y>0$ as $\lambda \to \infty$. Since φ is regularly varying at 0 with exponent $\alpha - 1$,

$$\frac{\varphi\left(c\lambda^{-1}\right)}{\varphi(\lambda^{-1})} \to c^{\alpha-1} = (\alpha-1)^{-1}\alpha^{\alpha} \quad \text{as } \lambda \to \infty.$$

and

$$\lim_{\lambda\to\infty} \frac{M^{-1}(\lambda x) - M^{-1}(\lambda)}{\varphi(\lambda^{-1})} = (\alpha-1)^{-1}\alpha^{\alpha}\left(1 - x^{-(\alpha-1)}\right) \tag{6.13}$$

follow. Then Lemma 11 below shows (6.3). □

Lemma 11. (6.13) *implies* (6.3).

Proof. Assume (6.13). Since $M^{-1}(x)$ has a monotone density

$$\left(M^{-1}(x)\right)' = \frac{1}{m(M^{-1}(x))},$$

the monotone density theorem implies

$$\lim_{\lambda\to\infty}\left(\frac{M^{-1}(\lambda x) - M^{-1}(\lambda)}{\varphi(\lambda^{-1})}\right)' = \left((\alpha-1)^{-1}\alpha^{\alpha}\left(1 - x^{-(\alpha-1)}\right)\right)',$$

which is

$$\lim_{\lambda\to\infty} \frac{\lambda}{\varphi(\lambda^{-1})\, m(M^{-1}(\lambda x))} = \alpha^{\alpha}x^{-\alpha}.$$

Setting $x=1$ and $u = M^{-1}(\lambda)$, we have

$$\lim_{u\to 0} \frac{M(u)}{\varphi(M(u)^{-1})\, m(u)} = \alpha^{\alpha}.$$

For any $\epsilon > 0$ there exists $\delta > 0$ such that for any $u \in (-\delta, 0)$

$$\alpha^{-\alpha} - \epsilon \le \frac{\varphi\left(M(u)^{-1}\right) m(u)}{M(u)} \le \alpha^{-\alpha} + \epsilon$$

are valid. Noting $m(u) = M(u)'$, we see

$$\left(\alpha^{-\alpha} - \epsilon\right)(-x) \le \int_x^0 \frac{\varphi\left(M(u)^{-1}\right)}{M(u)} dM(u) \le \left(\alpha^{-\alpha} + \epsilon\right)(-x)$$

for any $x \in (-\delta, 0)$, hence

$$\left(\alpha^{-\alpha} - \epsilon\right)(-x) \leq \int_{M(x)}^{\infty} \frac{\varphi\left(y^{-1}\right)}{y} dy = \int_{0}^{M(x)^{-1}} \frac{\varphi(z)}{z} dz \leq \left(\alpha^{-\alpha} + \epsilon\right)(-x).$$

Since $\varphi(z)/z$ is a regularly varying function at 0 with exponent $\alpha - 2$,

$$\int_{0}^{y} \frac{\varphi(z)}{z} dz \sim \frac{\varphi(y)}{\alpha - 1} \quad \text{as } y \downarrow 0$$

is valid, which implies

$$\frac{\varphi\left(M(x)^{-1}\right)}{\alpha - 1} \sim \alpha^{-\alpha}(-x) \quad \text{as } x \uparrow 0,$$

hence

$$M(x) \sim \frac{\beta^{\beta}}{\beta - 1} \varphi^{-1}(-x)^{-1} \quad \text{as } x \uparrow 0.$$

This is equivalent to (6.3). □

In the above two propositions we stated the conditions which should be satisfied by $m \in \mathcal{E}$ in terms of its spectral function σ. It may be preferable to describe the result by m itself directly. To do so, unfortunately we have to impose a more restrictive condition on m, and combining Proposition 1 and Proposition 2 we have

Theorem 7. *Let $\alpha \geq 2$, $k > \alpha - 1$ and φ is a regularly varying function at 0 with exponent $\alpha - 1$. Let $m \in \mathcal{E}$ be a non-decreasing function with $l = 0$ and $M(0) = \infty$. Assume m satisfies*

$$\int_{-\infty}^{-1} M(x)^{k} dx < \infty.$$

Then, the property

$$p(t) \sim \frac{\alpha^{2\alpha}}{\Gamma(1+\alpha)} \frac{1}{t} \varphi\left(\frac{1}{t}\right) \quad as \quad t \to \infty$$

holds if and only if the asymptotics below is valid.

$$m(x) \sim \frac{\beta^{\beta}}{-x\varphi^{-1}(-x)} \quad as \quad x \uparrow 0.$$

Proof. The proof is immediate from the above two propositions if we observe $\phi(t) = t^{k}$ satisfies all the requirements needed in Proposition 2. □

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